

ON THE HOT SPOTS OF A CATALYTIC SUPER-BROWNIAN MOTION

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ABSTRACT. Consider the catalytic super-Brownian motion X^e (reactant) in \mathbb{R}^d , $d \leq 3$, which branching rates vary randomly in time and space and in fact are given by an ordinary super-Brownian motion ϱ (catalyst). Our main object of study is the collision local time $L = L_{[\varrho, X^e]}(d[s, x])$ of catalyst and reactant. It determines the covariance measure in the martingale problem for X^e and reflects the occurrence of “hot spots” of reactant which can be seen in simulations of X^e . In dimension 2, spatial marginal collision densities exist and, via self-similarity, enter as factor in the long-term random ergodic limit of L (diffusiveness of the 2-dimensional model).

1. INTRODUCTION

The ordinary *super-Brownian motion* $\varrho = (\varrho_t, t \geq 0)$ in Euclidean space \mathbb{R}^d can be obtained as a limit of branching particle systems. In this branching particle system, the particles evolve according to independent Brownian motions in \mathbb{R}^d , and additionally, with constant rate $\gamma > 0$, each particle splits independently into 2 or 0 particles with equal probability (this is a critical binary branching mechanism).

We now interpret ϱ as a *catalyst process*: $\varrho_t(dx)$ is the amount of catalytic “particles” at time t in the volume element dx of \mathbb{R}^d . We then let a super-Brownian motion $X^e = (X_t^e, t \geq 0)$ evolve in this *catalytic random medium* ϱ . Intuitively X^e describes *reactant* “particles” which are evolving according to independent Brownian motions and which are performing a critical binary branching, but at random time-space varying rates given by ϱ . In fact, the rate of branching of an intrinsic reactant particle with Brownian path W is controlled by the *collision local time* $L_{[\varrho, W]}$ of ϱ and W , defined as the measure

$$L_{[\varrho, W]}(ds) := \lim_{\varepsilon \downarrow 0} ds \int \varrho_s(dy) p(\varepsilon, y - W_s),$$

where p is the standard heat kernel $p(t, x) = [2\pi t]^{-d/2} \exp[-|x|^2/2t]$, $(t, x) \in (0, \infty) \times \mathbb{R}^d$. According to [BEP91], this collision local time $L_{[\varrho, W]}$ makes sense non-trivially in dimension $d \leq 3$, and vanishes for $d \geq 4$ (where the Brownian reactant particles do not hit the catalyst ϱ). Thus we restrict our attention to $d \leq 3$ (since otherwise X^e degenerates to the heat flow).

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The *catalytic super-Brownian motion* X^ϱ was constructed in [DF97a]. Let ϱ and X^ϱ start at time 0 with Lebesgue measures ℓ_c and ℓ_r , respectively. From the papers [DF97a, DF97b, EF98, FK97] it is known that X_T^ϱ converges in law as $T \uparrow \infty$ to a limit X_∞^ϱ with full expectation ℓ_r (*persistence*)¹⁾. The approach of [FK97] to solve the most difficult case, namely convergence in the critical dimension $d = 2$, was to study the local structure of X^ϱ and then to use a self-similarity argument. In fact, they showed that in dimensions $d = 2, 3$, given the catalyst ϱ , the reactant X^ϱ has a *density field* ξ^ϱ :

$$X_t^\varrho(dx) = \xi_t^\varrho(x)dx, \quad t > 0.$$

Moreover, off the time-space support of the catalyst ϱ (which is a Lebesgue zero set), ξ^ϱ can be chosen as a (time-space) C^∞ -function that solves the heat equation, just as intuitively expected.

Simulations of (ϱ, X^ϱ) in dimension $d = 2$ (see the figure in [FK97]) confirm the *heuristic picture* one has. Namely, at late times T ,

- the reactant X_T^ϱ is rather uniform outside of the catalyst ϱ_T ,
- it is absent inside of the clumps of ϱ_T (since a huge rate of branching causes mainly killing),
- but occasionally also some *hot spots* of the reactant occur in the *interface* of ϱ_T and X_T^ϱ , that is in the boundary region of the catalytic clumps.

According to [FK97], in the two-dimensional case, the (local) long-term limit X_∞^ϱ is in fact a random multiple of Lebesgue measure [the factor is given by $\xi_1^\varrho(0)$]. But so far the investigations on the catalytic super-Brownian motion X^ϱ do not reflect anything on the hot spots seen in the pictures. Our approach to gain some information about them is to study the *collision local time* $L := L_{[\varrho, X^\varrho]}$ of ϱ and X^ϱ defined as the limit of

$$(1) \quad L_\varepsilon(d[s, x]) := ds \, \varrho_s(dx) \int X_s^\varrho(dy) p(\varepsilon, x - y),$$

as $\varepsilon \downarrow 0$.

Actually there is a further motivation to study this collision local time $L_{[\varrho, X^\varrho]}$. It occurs indeed in the description of the *martingale problem* for the process X^ϱ (see Corollary 4 below). For martingale problems of catalytic super-Brownian motions, see also [DF94, Del96, Led97].

Let us present the results. We prove that in all dimensions of non-trivial existence of X^ϱ the collision local time L of catalyst and reactant makes non-trivially sense (see Theorem 3 below). [In dimension 1, it is known that both ϱ_s and X_s^ϱ are absolutely continuous (cf. [KS88] and [DFR91], respectively); thus L_ε and hence L simplify in this case.] This non-trivial existence of L reflects the high fluctuations of X^ϱ in the interface of catalyst and reactant, seen as hot spots in simulations. Our *main result* however is that for $d = 2$ the *marginal measure* of $L = L_{[\varrho, X^\varrho]}$ concerning the space variable is *absolutely continuous* (Theorem 5). Note that this is in contrast, for instance, with the (one-dimensional) single-point catalytic model of [DF94], say X^{δ_0} , where, together with the catalyst δ_0 , the space marginal of the collision local time $L_{[\delta_0, X^{\delta_0}]}$ is concentrated in the single space point 0, hence is singular (even atomic). Finally, in dimension 2, using the *self-similarity* of $L_{[\varrho, X^\varrho]}$ which follows from the

¹⁾ In the three-dimensional case, for simplification it was assumed in [DF97b] that the catalyst process ϱ is already in its corresponding equilibrium.

self-similarity of (ϱ, X^ϱ) , we show that $T^{-1}L([0, T] \times (\cdot))$ has a *random ergodic limit* as $T \uparrow \infty$, which indicates *diffusive* features in the long-term behavior in $d = 2$.

It remains *open* whether also in dimension 3 space-marginal collision densities exist since our L^2 -approach fails in this case (see Remark 6 below).

The *outline* of the paper is as follows. In Section 2 we introduce formal definitions of the processes ϱ and X^ϱ and state the results. The following two sections are then devoted to the proofs of our two theorems. In an appendix we collect some results on ordinary and catalytic super-Brownian motions used in the proofs.

2. STATEMENT OF RESULTS

2.1. Notation. The lower index $+$ on a set will always refer to the collection of all its nonnegative members. Similarly, f_+ is the nonnegative part of f . The supremum norm is denoted by $\|\cdot\|_\infty$. Let c always refer to a (finite) constant whose value may vary from place to place.

We denote by $\mathcal{B}(E)$ the space of all real Borel measurable functions defined on a polish space E . We also denote by $\mathcal{B}(E)$ the Borel σ -field of E .

For a fixed constant $q > d$, introduce the reference function $\phi_q \in \mathcal{B}(\mathbb{R}^d)$:

$$(2) \quad \phi_q(x) := [1 + |x|^2]^{-q/2}, \quad x \in \mathbb{R}^d.$$

Set $\mathcal{B}^q := \{f \in \mathcal{B}(\mathbb{R}^d); \|f/\phi_q\|_\infty < \infty\}$. Let $C_c(\mathbb{R}^d)$ denote the collection of all continuous functions on \mathbb{R}^d with compact support.

If ν is a Radon measure on \mathbb{R}^d , we write (ν, f) for $\int \nu(dx) f(x)$ (if the integral makes sense). Let \mathcal{M}_q denote the set of all Radon measures ν on \mathbb{R}^d such that $(\nu, \phi_q) < \infty$. This space of tempered measures is endowed with the coarsest topology such that the maps $\nu \mapsto (\nu, f)$ are continuous for $f \in C_c(\mathbb{R}^d) \cup \{\phi_q\}$, getting a Polish space. Since $q > d$, Lebesgue measure belongs to \mathcal{M}_q .

We consider the polish space $\mathcal{C} := C(\mathbb{R}_+, \mathcal{M}_q)$ of all continuous functions from \mathbb{R}_+ to \mathcal{M}_q equipped with the topology of uniform convergence on compacta.

Let $(P_t, t \geq 0)$ denote the semigroup of heat flow on \mathbb{R}^d :

$$(3) \quad P_t[f](x) := \int dy p(t, x - y) f(y), \quad t > 0, \quad f \in \mathcal{B}_+(\mathbb{R}^d).$$

2.2. Catalyst and reactant process. We start by introducing the catalyst process.

Definition 1 (catalyst process). Let $\gamma > 0$ and $\nu \in \mathcal{M}_q$. There exists a unique probability measure \mathbb{P}_ν on $(\mathcal{C}, \mathcal{B}(\mathcal{C}))$, such that the coordinate process $\varrho = (\varrho_t, t \geq 0)$ on \mathcal{C} is a super-Brownian motion with constant branching rate γ and starting measure ν . That is, ϱ is a continuous time-homogeneous strong Markov process with the following properties:

- \mathbb{P}_ν -almost surely, $\varrho_0 = \nu$,
- for every $f \in \mathcal{B}_+^q$, $t \geq r \geq 0$, we have²⁾

$$(4) \quad \mathbb{E}_\nu \left[e^{-(\varrho_t, f)} \mid \sigma(\varrho_s, s \in [0, r]) \right] = e^{-(\varrho_r, w(t-r))},$$

²⁾ We use the following convention: If \mathcal{P} is a probability law, then the corresponding letter \mathcal{E} refers to the related expectation symbol.

where w is the unique nonnegative solution on $\mathbb{R}_+ \times \mathbb{R}^d$ of the log-Laplace equation

$$(5) \quad w(t, x) + \gamma \int_0^t ds P_s[w^2(t-s)](x) = P_t[f](x).$$

We write \mathbb{P} for \mathbb{P}_ν in the case $\nu = i_c \ell$, where $i_c > 0$ and ℓ is the (normalized) Lebesgue measure on \mathbb{R}^d . \diamond

From now on we assume that $d \leq 3$, and that ϱ is distributed³⁾ according to \mathbb{P} . Next we recall the definition of the catalytic super-Brownian motion X^ϱ in the random medium ϱ (see [DF97a] for details).

Definition 2 (catalytic super-Brownian motion). Fix $(r, \mu) \in \mathbb{R}_+ \times \mathcal{M}_q$ and $\kappa > 0$. For convenience, set $\mathcal{C}' := C([r, \infty), \mathcal{M}_q)$. There exists a (measurable) probability kernel $\varrho \mapsto \mathbb{P}_{r, \mu}^\varrho$ from $(\mathcal{C}, \mathcal{B}(\mathcal{C}))$ to $(\mathcal{C}', \mathcal{B}(\mathcal{C}'))$ such that the coordinate process $X^\varrho = (X_t^\varrho, t \geq r)$ on \mathcal{C}' is a *super-Brownian motion in the catalytic medium ϱ* . That is, \mathbb{P} -a.s. under $\mathbb{P}_{r, \mu}^\varrho$, the process X^ϱ is continuous time-inhomogeneous Markov with the following properties:

- $\mathbb{P}_{r, \mu}^\varrho$ -almost surely, $X_r^\varrho = \mu$,
- for every $f \in \mathcal{B}_+^q$, $t \geq s \geq r$, we have

$$(6) \quad \mathbb{E}_{r, \mu}^\varrho \left[e^{-(X_t^\varrho, f)} \mid \sigma(X_u^\varrho, u \in [r, s]) \right] = e^{-(X_s^\varrho, v_t(s))},$$

where v_t is the unique nonnegative solution on $[r, \infty) \times \mathbb{R}^d$ of the catalytic log-Laplace equation

$$(7) \quad v(s, x) + \kappa \int_s^\infty du \int \varrho_u(dy) p(u-s, x-y) v^2(u, y) = J(s, x),$$

with $J(s) := \mathbf{1}_{t \geq s} P_{t-s}[f]$.

Often, we also pass from the quenched distributions $\mathbb{P}_{r, \mu}^\varrho$ to the annealed law $\mathbb{E}[\mathbb{P}_{r, \mu}^\varrho]$. \diamond

2.3. Existence of collision local time of catalyst and reactant. For our constant $q > d$, we introduce the function space $H^q := \bigcup_{T \geq 0} H_T^q$, where

$$(8) \quad H_T^q := \left\{ g \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d); \text{supp } g \subset [0, T] \times \mathbb{R}^d, \|g/\phi_q\|_\infty < \infty \right\},$$

with $\|g/\phi_q\|_\infty = \sup_{(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d} |g(s, x)|/\phi_q(x)$, and $\text{supp } g$ denoting the support of g .

Recall the approximated collision local time L_ε of ϱ and X^ϱ introduced already in (1). We are now ready to state our first result, the existence of the collision local time $L = L_{[\varrho, X^\varrho]}$ of ϱ and X^ϱ . Recall that $d \leq 3$ and $(r, \mu) \in \mathbb{R}_+ \times \mathcal{M}_q$.

Theorem 3 (collision local time). *There exists a random variable denoted by $L = L_{[\varrho, X^\varrho]}$ defined on $(\mathcal{C} \times \mathcal{C}', \mathcal{B}(\mathcal{C} \times \mathcal{C}'))$, taking values in the set of Radon measures on $[r, \infty) \times \mathbb{R}^d$ with the following properties:*

- (i) **(tempered measure):** For every $T \geq r$, we have $\mathbb{E}[\mathbb{P}_{r, \mu}^\varrho(L, \mathbf{1}_{[r, T]} \phi_q)] < \infty$.
- (ii) **(existence via convergence):** For every $\varphi \in H^{2q}$,

$$\lim_{\varepsilon \downarrow 0} (L_\varepsilon, \varphi) = (L, \varphi), \quad \mathbb{E}[\mathbb{P}_{r, \mu}^\varrho] \text{-a.s.}$$

³⁾ In [FK97] more generally a class of so-called η -diffusive measures ν is introduced which allow that ϱ under \mathbb{P}_ν may serve as the catalyst for X^ϱ .

(iii) **(regularity):** For every $\varphi \in H^{2q}$, and $\mathbb{E}[\mathbb{P}_{r,\mu}^\varrho]$ -a.s., the process $((L, \mathbf{1}_{[r,t]}\varphi), t \geq r)$ is continuous and adapted to the filtration

$$(\mathcal{F}_t := \sigma(\varrho) \vee \sigma(X_s^\varrho, s \in [r, t]), t \geq r).$$

(iv) **(moments):** For every $m \geq 1$, $\varphi \in H^{2q}$, \mathbb{P} -a.s.,

$$(9) \quad \mathbb{E}_{r,\mu}^\varrho \left[\left[\int_{[r,\infty) \times \mathbb{R}^d} L(d[s, x]) \varphi(s, x) \right]^m \right] = m! \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{n_1, \dots, n_k \geq 1, \\ n_1 + \dots + n_k = m}} \prod_{i=1}^k (\mu, \chi_{n_i}(r)),$$

where the functions χ_n , $n \geq 1$, belong to H^q and are recursively defined by

$$(10) \quad \chi_n(s, x) := \kappa \int_s^\infty du \int \varrho_u(dy) p(u - s, x - y) \left[\sum_{i=1}^{n-1} \chi_i(u, y) \chi_{n-i}(u, y) \right], \quad n \geq 2,$$

with the initial condition

$$(11) \quad \chi_1(s, x) := \int_s^\infty du \int \varrho_u(dy) p(u - s, x - y) \varphi(u, y), \quad (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Consequently, in dimensions $d \leq 3$, the collision local time $L = L_{[\varrho, X^\varrho]}$ of catalyst and reactant exists non-trivially, reflecting in particular the occurrence of hot spots in the mentioned 2-dimensional simulations.

The proof of this theorem is postponed to Section 3.

As an *application*, we can now describe the covariance measure of the martingale measure associated with X^ϱ . Let $C_b^{1,2}$ denote the set of bounded functions $\varphi \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that the partial derivatives $\frac{\partial \varphi}{\partial s}$ and $\frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ exist and are continuous and bounded. It is easy to check that under $\mathbb{E}[\mathbb{P}_{r,\mu}^\varrho]$ the process $(M\varphi_{r,t}, t \geq r)$ defined by

$$(12) \quad M\varphi_{r,t} := (X_t^\varrho, \varphi(t)) - (X_r^\varrho, \varphi(r)) - \int_r^t ds \left(X_s^\varrho, \frac{\partial \varphi}{\partial s}(s) + \frac{1}{2} \Delta \varphi(s) \right),$$

is an $(\mathcal{F}_t, t \geq r)$ -martingale [note that $\mathcal{F}_r = \sigma(\varrho) \vee \sigma(X_r)$]. Thanks to the Markov property of X^ϱ (given ϱ), and the moment formula (A.9) for X^ϱ stated in the appendix, we get that for $\varphi, \psi \in C_b^{1,2}$, \mathbb{P} -a.s. for all $s \geq r$ and $t \geq r$,

$$(13) \quad \mathbb{E}_{r,\mu}^\varrho [M\varphi_{r,s} M\psi_{r,t}] = 2\kappa \int \mu(dx) \int_r^{s \wedge t} du \int \varrho_u(dy) p(u - r, x - y) \varphi(u, y) \psi(u, y).$$

The functional $M : \varphi \mapsto M\varphi$ defined on $C_b^{1,2}$ can be extended to an *orthogonal martingale measure* on H^q . Let $\langle M \rangle$ denote its *covariance measure*. Now we show how $\langle M \rangle$ can be expressed in terms of the collision local time $L = L_{[\varrho, X^\varrho]}$. Recall that $d \leq 3$ and that $(r, \mu) \in \mathbb{R}_+ \times \mathcal{M}_q$.

Corollary 4 (covariance measure). *For every $\varphi \in H^q$, $\mathbb{E}[\mathbb{P}_{r,\mu}^\varrho]$ -a.s. for every $t \geq r$, we have*

$$(14) \quad \langle M\varphi \rangle_{r,t} = 2\kappa \int_{[r,t] \times \mathbb{R}^d} L(d[s, y]) \varphi^2(s, y).$$

Proof. Using the Markov property of X^ϱ (given ϱ) and an obvious extension of the second moment formula (13), we obtain for $\varphi \in H^q$, \mathbb{P} -a.s. for all $t \geq s \geq r$,

$$\mathbb{E} \mathbb{E}_{r,\mu}^\varrho [(M\varphi_{r,t})^2 \mid \mathcal{F}_s] = (M\varphi_{r,s})^2 + 2\kappa \int_s^t X_s^\varrho(dx) \int_s^t du \int \varrho_u(dy) p(u-s, x-y) \varphi^2(u, y).$$

Since

$$\left(\int_{[r,t] \times \mathbb{R}^d} L(d[s, y]) \varphi^2(s, y), t \geq r \right)$$

is in t non-decreasing and continuous, is adapted to $(\mathcal{F}_t, t \geq r)$, and zero for $t = r$, we get that

$$\left(\langle M\varphi \rangle_{r,t} - 2\kappa \int_{[r,t] \times \mathbb{R}^d} L(d[s, y]) \varphi^2(s, y), t \geq r \right)$$

is a continuous martingale under $\mathbb{E}[\mathbb{P}_{r,\mu}^\varrho]$ with bounded variation starting at time $t = r$ from 0. This martingale is then constant and, in fact, equal to 0, giving the claim (14). \square

2.4. Collision local time in dimension two. We now state our results for the collision local time in the “critical” dimension $d = 2$. For convenience, we introduce the following abbreviation for an annealed law:

$$\mathbf{P} := \mathbb{E}[\mathbb{P}_{0,i_r\ell}] = \mathbb{E}_{i_c\ell}[\mathbb{P}_{0,i_r\ell}], \quad \text{where } i_r > 0.$$

(That is, we now focus on the situation $r = 0$ and $\mu = i_r\ell$.)

Theorem 5 (two-dimensional collision local time). *Let $d = 2$.*

(a) (local spatial L^2 collision densities): *For every $t \geq s \geq 0$ and $z \in \mathbb{R}^2$,*

$$\left(\int_{[s,t] \times \mathbb{R}^d} L(d[r, y]) p(\varepsilon, z - y), \varepsilon > 0 \right)$$

converges in $L^2(\mathbf{P})$ as $\varepsilon \downarrow 0$ to a random variable denoted by $\lambda_{[s,t]}(z)$. It has expectation

$$\mathbf{E}[\lambda_{[s,t]}(z)] = i_c i_r (t - s),$$

and its finite variance is non-zero provided that $s < t$.

(b) (spatial absolute continuity): *For $t \geq s \geq 0$, there exists a measurable version of $\lambda_{[s,t]}$ with respect to $\mathcal{B}(\mathbb{R}^2) \times \mathcal{F}_t$, and \mathbf{P} -a.s. the measure $L([s, t] \times (\cdot))$ on \mathbb{R}^2 is absolutely continuous and can be represented as*

$$L([s, t] \times dx) = \lambda_{[s,t]}(x) dx.$$

(c) (self-similarity): *Under \mathbf{P} , the laws of the scaled collision local times*

$$K^{-2} L(K(\cdot) \times K^{1/2}(\cdot))$$

are independent of the scaling factor $K > 0$.

(d) (random ergodic limit): *The following convergence in \mathcal{M}_q holds in law with respect to \mathbf{P} :*

$$\lim_{T \uparrow \infty} T^{-1} L([0, T] \times (\cdot)) = \lambda_{[0,1]}(0) \ell$$

(with ℓ the Lebesgue measure and $0 < \mathbf{Var}[\lambda_{[0,1]}(0)] < \infty$).

Consequently, in dimension 2, the spatial marginal measures $L([s, t] \times (\cdot))$ of the collision local time $L_{[\varrho, X^e]}$ of catalyst and reactant have non-degenerated densities $\lambda_{[s, t]}(z)$ (provided that $s < t$). Moreover, $\lambda_{[0, 1]}(0)$ enters as random factor of Lebesgue measure in the long-term ergodic limit. Recall that this reflects the diffusive features of the hot spots.

Remark 6 (dimension three). The $L^2(\mathbf{P})$ -convergence in part (a) does *not* hold for $d = 3$. In fact, in the three-dimensional case an infinite term would be involved in our calculations, see the remark following (29) in the proof below. Recall on the other hand that in dimension one, $L_{[\varrho, X^e]}$ should be rather “regular”. \diamond

Remark 7 (regularity). It is an *open problem* whether the spatial collision density functions $\lambda_{[s, t]}$ have some regularities properties in the space variable. Note also that the exceptional set in the \mathbf{P} -a.s. statement in (b) depends on $[s, t]$. One would also like to know whether this situation can be improved. \diamond

The statement (c) follows from the self-similarity of (ϱ, X^e) by standard arguments (compare with [DF97b, Subsections 4.1 and 4.2]). Otherwise the proof of Theorem 5 will be provided in Section 4.

3. EXISTENCE OF COLLISION LOCAL TIME (PROOF OF THEOREM 3)

Recall that $d \leq 3$. First of all we state the following lemma.

Lemma 8 (approximated moment increments). *For every $m \geq 1$, $r \geq 0$, $\mu \in \mathcal{M}_q$, $T \geq 0$, $\xi \in (0, 1/4)$, \mathbb{P} -a.s. there exists a finite constant M_m (depending on ϱ) such that for every $\varphi \in H_T^{2q}$, $t' \geq t \geq 0$, $1 \geq \varepsilon' \geq \varepsilon > 0$,*

$$(15) \quad \mathbb{E}_{r, \mu}^{\varrho} [(L_{\varepsilon}, \varphi \mathbf{1}_{[t, t']})^{2m}] \leq M_m \|\varphi / \phi_{2q}\|_{\infty}^{2m} \left[|t - t'|^{\xi} \left(1 + \log_+ (1/|t - t'|) \right) \right]^{2m},$$

$$(16) \quad \mathbb{E}_{r, \mu}^{\varrho} \left[[(L_{\varepsilon}, \varphi) - (L_{\varepsilon'}, \varphi)]^{2m} \right] \leq M_m \|\varphi / \phi_{2q}\|_{\infty}^{2m} \left[|\varepsilon - \varepsilon'|^{\xi} \left(1 + \log_+ (1/|\varepsilon - \varepsilon'|) \right) \right]^{2m}.$$

Based on this lemma, the proof of Theorem 3 (ii) and (iii) are similar to the proof of Proposition 5.1 based on Lemma 5.2 in [Del96] with the obvious changes and is left to the reader. (iv) is not stated in Proposition 5.1 there, but it is a by-product of its proof [take the limit in (32)]. Eventually, (i) is proved by using the monotone convergence theorem with the moment formula (9) and (A.2) (in the appendix) with $m = 1$ and the inequality (A.1).

Proof of Lemma 8. Fix $\mu \in \mathcal{M}_q$, $\xi \in (0, 1/4)$, and $T \geq r \geq 0$ (otherwise the moments disappear). We will verify (15); the proof of (16) is similar and is left to the reader.

Note first that for fixed $\varepsilon > 0$,

$$(17) \quad \sup_{x \in \mathbb{R}^d, y \in \mathbb{R}^d} \frac{\phi_q(y) p(\varepsilon, x - y)}{\phi_q(x)} < \infty.$$

Let $\varphi \in H_T^{2q}$. Since ϱ is \mathbb{P} -a.s. a continuous \mathcal{M}_q -valued path, it is then clear that the functions $(s, x) \mapsto \int \varrho_s(dy) p(\varepsilon, x - y) \varphi(s, y)$ belong to H_T^q . Thanks to the remarks at the beginning of Subsection A.1, we see that, for fixed t, t', ε , the functions

$$(18) \quad (s, x) \mapsto J_{\varepsilon}(s, x) := \int_s^{\infty} du \int dz p(u - s, x - z) \int \varrho_u(dy) p(\varepsilon, z - y) \varphi(u, y) \mathbf{1}_{[t, t']}(u)$$

are well-defined and belong to H_T^q .

We will now prove that \mathbb{P} -a.s. there exists a finite constant c such that for every $\varphi \in H_T^{2q}$, $t' \geq t \geq 0$, $1 \geq \varepsilon > 0$,

$$(19) \quad |J_\varepsilon(s, x)| \leq c \mathbf{1}_{[0, T]}(s) \phi_q(x) \|\varphi / \phi_{2q}\|_\infty \left[|t - t'|^\xi \left(1 + \log_+ (1/|t - t'|) \right) \right].$$

Clearly $|J_\varepsilon(s, x)| / \|\varphi / \phi_{2q}\|_\infty$ is bounded from above by

$$K_1 := \mathbf{1}_{[0, T]}(s) \int_s^T du \int \varrho_u(dy) p(u - s + \varepsilon, x - y) \phi_{2q}(y) \mathbf{1}_{[t, t']}(u).$$

We assume that $T \geq t$ (otherwise $K_1 = 0$). Introduce the quantity

$$K_2 := \mathbf{1}_{[0, T \wedge t']}(s) \int_{s \vee t}^{T \wedge t'} du \int \varrho_u(dy) p(u - s \vee t, x - y) \phi_{2q}(y).$$

Thanks to (A.6), we have $K_2 \leq \mathbf{1}_{[0, T]}(s) C_2 |t - t'|^\xi \phi_q(x)$. Now

$$|K_1 - K_2| \leq \mathbf{1}_{[0, T \wedge t']}(s) \int_{s \vee t}^{T \wedge t'} du \int \varrho_u(dy) |p(u - s + \varepsilon, x - y) - p(u - s \vee t, x - y)| \phi_{2q}(y).$$

Using the inequality

$$(20) \quad |p(v_1, z) - p(v_2, z)| \leq c \int_{v_1}^{v_2} dv v^{-1} p(2v, z),$$

where the constant c is independent of $z \in \mathbb{R}^d$ and $v_2 \geq v_1 > 0$, we get that

$$\begin{aligned} |K_1 - K_2| &\leq c \mathbf{1}_{[0, T \wedge t']}(s) \int_{s \vee t}^{T \wedge t'} du \int \varrho_u(dy) \phi_{2q}(y) \int_{u - s \vee t}^{u - s + \varepsilon} dv v^{-1} p(2v, x - y) \\ &= c \mathbf{1}_{[0, T \wedge t']}(s) \int_0^{T \wedge t' - s + \varepsilon} dv v^{-1} \int_{s \vee t \vee (v + s - \varepsilon)}^{T \wedge t' \wedge (v + s \vee t)} du \int \varrho_u(dy) \phi_q^2(y) p(2v, x - y). \end{aligned}$$

In view of (A.5) and (A.1), we may continue with

$$\leq c \mathbf{1}_{[0, T \wedge t']}(s) \phi_q(x) \int_0^{T \wedge t' - s + \varepsilon} dv v^{-1} |T \wedge t' \wedge (v + s \vee t) - s \vee t \vee (v + s - \varepsilon)|^\xi,$$

where c is independent of t', t, ε, x . It is easy to check that

$$(21) \quad \begin{aligned} &\int_0^{T \wedge t' - s + \varepsilon} dv v^{-1} |T \wedge t' \wedge (v + s \vee t) - s \vee t \vee (v + s - \varepsilon)|^\xi \\ &\leq c |t' - t|^\xi \left(1 + \log_+ (1/|t' - t|) \right), \end{aligned}$$

where c is independent of t', t and ε . As a conclusion we obtain (19).

Using the estimate (A.6), a straight forward induction shows that all the functions χ_n , $n \geq 1$, of the recurrence relation (10) with initial condition $\chi_1 = J_\varepsilon$ belong to H_T^q and satisfy

$$|\chi_n(s, x)| \leq c \mathbf{1}_{[0, T]}(s) \phi_q(x) \|\varphi / \phi_{2q}\|_\infty^n \left[|t - t'|^\xi \left(1 + \log_+ (1/|t - t'|) \right) \right]^n.$$

(Note that c is independent of φ, t, t' and ε .) Then the claim (15) is a consequence of (A.9) with $f = 0$ and

$$(22) \quad g(s, z) := \int \varrho_s(dy) p(\varepsilon, z - y) \varphi(s, y) \mathbf{1}_{[t, t']}(s),$$

finishing the proof. \square

4. TWO-DIMENSIONAL COLLISION LOCAL TIME (PROOF OF THEOREM 5)

We now assume that $d = 2$.

4.1. Local spatial collision densities [proof of (a)]. For the claimed L^2 -convergence, it is enough to check that, for fixed s, t, z ,

$$(23) \quad J^{\varepsilon, \varepsilon'} := \mathbf{E} \left[\int_{[s, t] \times \mathbb{R}^2} L(d[r, y]) p(\varepsilon, z - y) \int_{[s, t] \times \mathbb{R}^2} L(d[r', y']) p(\varepsilon', z - y') \right]$$

converges in \mathbb{R}_+ as ε and ε' decrease to 0.

For $f \in L_+^1(\mathbb{R}^2)$ with $\int dx f(x) = 1$, and $\varepsilon > 0, z \in \mathbb{R}^2$, we set

$$(24) \quad f_{\varepsilon, z}(x) := \varepsilon^{-1} f\left(\varepsilon^{-1/2}(x - z)\right).$$

Note that $f_{\varepsilon, z}(x)dx$ converges weakly to $\delta_z(dx)$, the Dirac mass at z , as ε decreases to 0. We will prove the following stronger result.

Lemma 9. *For fixed $s, t, z, z', f, f' \in L_+^1(\mathbb{R}^2)$ such that $\int dx f(x) = 1 = \int dx f'(x)$, the well-defined quantity*

$$J^{\varepsilon, \varepsilon'}(z, z') := \mathbf{E} \left[\int_{[s, t] \times \mathbb{R}^2} L(d[r, y]) f_{\varepsilon, z}(y) \int_{[s, t] \times \mathbb{R}^2} L(d[r', y']) f'_{\varepsilon', z'}(y') \right]$$

converges to a finite limit independent of f, f' , as ε and ε' decrease to 0.

Note that we need the convergence for $z = z'$ to prove (23) and then (a). Note also that although f and f' are not in \mathcal{B}^q a priori, we show that $J^{\varepsilon, \varepsilon'}$ is well-defined.

Proof of Lemma 9. By a standard monotone class argument, we deduce from the quenched moment formula (9) for collision local time with $m = 2$, that for $g \in \mathcal{B}_+((\mathbb{R}_+)^2 \times (\mathbb{R}^2)^2)$,

$$\begin{aligned} & \mathbf{E} \left[\int_{\mathbb{R}_+ \times \mathbb{R}^2} L(d[r, y]) \int_{\mathbb{R}_+ \times \mathbb{R}^2} L(d[r', y']) g(r, r', y, y') \right] \\ &= \mathbb{E} \left[2 i_r \kappa \int dx \int_0^\infty ds_1 \int \varrho_{s_1}(dy_1) p(s_1, y_1 - x) \int_{s_1}^\infty ds_2 \int \varrho_{s_2}(dy_2) p(s_2 - s_1, y_2 - y_1) \right. \\ & \quad \left. \int_{s_1}^\infty ds_3 \int \varrho_{s_3}(dy_3) p(s_3 - s_1, y_3 - y_1) g(s_2, s_3, y_2, y_3) \right. \\ & \quad \left. + i_r^2 \int dx_1 \int_0^\infty ds_1 \int \varrho_{s_1}(dy_1) p(s_1, y_1 - x_1) \right. \\ & \quad \left. \int dx_2 \int_0^\infty ds_2 \int \varrho_{s_2}(dy_2) p(s_2, y_2 - x_2) g(s_1, s_2, y_1, y_2) \right]. \end{aligned}$$

Thus we can write

$$(25) \quad J^{\varepsilon, \varepsilon'} = 2 i_r \kappa J_1^{\varepsilon, \varepsilon'} + i_r^2 J_2^{\varepsilon, \varepsilon'},$$

where

$$J_1^{\varepsilon, \varepsilon'}(z, z') := \int_0^t ds_1 \mathbb{E} \left[\int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \int \varrho_{s_1}(dy_1) \int \varrho_{s_2}(dy_2) \int \varrho_{s_3}(dy_3) \right. \\ \left. p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) \right]$$

and

$$J_2^{\varepsilon, \varepsilon'}(z, z') := \mathbb{E} \left[\int_s^t ds_1 \int_s^t ds_2 \int \varrho_{s_1}(dy_1) \int \varrho_{s_2}(dy_2) f_{\varepsilon, z}(y_1) f'_{\varepsilon', z'}(y_2) \right]$$

are third and second moment expressions of the catalyst process only, respectively. We easily compute $J_2^{\varepsilon, \varepsilon'}$ thanks to the moment formula (A.2) for ordinary super-Brownian motion (with $f = 0$ and g properly chosen):

$$\begin{aligned} J_2^{\varepsilon, \varepsilon'}(z, z') &= 2\gamma i_c \int dx \int_0^t ds_3 \int_{s_3 \vee s}^t ds_1 \int_{s_3 \vee s}^t ds_2 \int dy_1 \int dy_2 \int dy_3 \\ &\quad p(s_3, y_3 - x) p(s_1 - s_3, y_1 - y_3) p(s_2 - s_3, y_2 - y_3) f_{\varepsilon, z}(y_1) f'_{\varepsilon', z'}(y_2) \\ &\quad + i_c^2 \int dx_1 \int dx_2 \int_s^t ds_1 \int_s^t ds_2 \int dy_1 \int dy_2 \\ &\quad p(s_1, y_1 - x_1) p(s_2, y_2 - x_2) f_{\varepsilon, z}(y_1) f'_{\varepsilon', z'}(y_2) \\ &= 2\gamma i_c \int dy_1 f_{\varepsilon, z}(y_1) \int dy_2 f'_{\varepsilon', z'}(y_2) \\ &\quad \int_0^t ds_3 \int_{s_3 \vee s}^t ds_1 \int_{s_3 \vee s}^t ds_2 p(s_1 + s_2 - 2s_3, y_1 - y_2) \\ &\quad + i_c^2 (t - s)^2 \\ &\leq 2\gamma i_c \int_0^t ds_3 \int_{s_3 \vee s}^t ds_1 \int_{s_3 \vee s}^t ds_2 p(s_1 + s_2 - 2s_3, 0) + i_c^2 (t - s)^2 =: K_2 < \infty. \end{aligned}$$

As $(\varepsilon, \varepsilon') \downarrow 0$, the quantity $J_2^{\varepsilon, \varepsilon'}(z, z')$ converges to

$$(26) \quad J_2^0(z, z') := 2\gamma i_c \int_0^t ds_3 \int_{s_3 \vee s}^t ds_1 \int_{s_3 \vee s}^t ds_2 p(s_1 + s_2 - 2s_3, z - z') + i_c^2 (t - s)^2 \leq K_2.$$

We can also compute $J_1^{\varepsilon, \varepsilon'}$ using the Markov property of ϱ at time s_1 and twice the moment formula (A.2):

$$\begin{aligned} J_1^{\varepsilon, \varepsilon'}(z, z') = & 2\gamma \int_0^t ds_1 \mathbb{E} \left[\int_{s_1}^t ds_4 \int_{s_4 \vee s}^t ds_2 \int_{s_4 \vee s}^t ds_3 \int \varrho_{s_1}(dy_1) \int \varrho_{s_1}(dy_5) \int dy_4 \int dy_2 \int dy_3 \right. \\ & p(s_4 - s_1, y_4 - y_5) p(s_2 - s_4, y_2 - y_4) p(s_3 - s_4, y_3 - y_4) \\ & \left. p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) \right] \\ & + \int_0^t ds_1 \mathbb{E} \left[\int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \int \varrho_{s_1}(dy_1) \int \varrho_{s_1}(dy_4) \int \varrho_{s_1}(dy_5) \int dy_2 \int dy_3 \right. \\ & p(s_2 - s_1, y_2 - y_4) p(s_3 - s_1, y_3 - y_5) p(s_2 - s_1, y_2 - y_1) \\ & \left. p(s_3 - s_1, y_3 - y_1) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) \right]. \end{aligned}$$

With obvious notation we write

$$(27) \quad J_1^{\varepsilon, \varepsilon'} = 2\gamma J_3^{\varepsilon, \varepsilon'} + J_4^{\varepsilon, \varepsilon'}.$$

Using again the moment formula, we get

$$(28) \quad J_3^{\varepsilon, \varepsilon'} = 2\gamma i_c J_5^{\varepsilon, \varepsilon'} + i_c^2 J_6^{\varepsilon, \varepsilon'},$$

where

$$\begin{aligned} J_5^{\varepsilon, \varepsilon'}(z, z') := & \int_0^t ds_1 \int_0^{s_1} ds_5 \int_{s_1}^t ds_4 \int_{s_4 \vee s}^t ds_2 \int_{s_4 \vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \int dy_5 \int dy_6 \\ & p(s_1 - s_5, y_1 - y_6) p(s_1 - s_5, y_5 - y_6) p(s_4 - s_1, y_4 - y_5) p(s_2 - s_4, y_2 - y_4) \\ & p(s_3 - s_4, y_3 - y_4) p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) \end{aligned}$$

and

$$\begin{aligned} J_6^{\varepsilon, \varepsilon'}(z, z') := & \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4 \vee s}^t ds_2 \int_{s_4 \vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \int dy_5 \\ & p(s_4 - s_1, y_4 - y_5) p(s_2 - s_4, y_2 - y_4) p(s_3 - s_4, y_3 - y_4) \\ & p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3). \end{aligned}$$

We now compute $J_6^{\varepsilon, \varepsilon'}$. Integrating over dy_1 , dy_5 , and dy_4 gives

$$\begin{aligned} J_6^{\varepsilon, \varepsilon'}(z, z') = & \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4 \vee s}^t ds_2 \int_{s_4 \vee s}^t ds_3 \int dy_2 \int dy_3 \\ & p(s_2 + s_3 - 2s_4, y_2 - y_3) p(s_2 + s_3 - 2s_1, y_2 - y_3) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3). \end{aligned}$$

The function

$$H_6(y_2, y_3) := \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4 \vee s}^t ds_2 \int_{s_4 \vee s}^t ds_3 p(s_2 + s_3 - 2s_4, y_2 - y_3) p(s_2 + s_3 - 2s_1, y_2 - y_3)$$

is continuous in (y_2, y_3) and bounded from above by $H_6(y, y) = K_6$ which is finite since $d = 2$. Thus $J_6^{\varepsilon, \varepsilon'}(z, z')$ is uniformly bounded by K_6 . Using that $f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) dy_2 dy_3$ converges

weakly to $\delta_z(dy_2)\delta_{z'}(dy_3)$, we deduce that $J_6^{\varepsilon,\varepsilon'}$ converges to

$$(29) \quad J_6^0(z, z') := H_6(z, z') \leq K_6.$$

Note that $H_6(z, z) = \infty$ if $d = 3$, which implies that $J_6^{\varepsilon,\varepsilon'}(z, z)$ doesn't converge for $d = 3$, however it is well-defined at least for $f(x) = f'(x) = p(1, x)$.

For $J_5^{\varepsilon,\varepsilon'}$ we get

$$\begin{aligned} J_5^{\varepsilon,\varepsilon'}(z, z') &= \int_0^t ds_1 \int_0^{s_1} ds_5 \int_{s_1}^t ds_4 \int_{s_4 \vee s}^t ds_2 \int_{s_4 \vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \\ &\quad p(s_1 + s_4 - 2s_5, y_1 - y_4) p(s_2 - s_4, y_2 - y_4) p(s_3 - s_4, y_3 - y_4) \\ &\quad p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) f_{\varepsilon,z}(y_2) f_{\varepsilon',z'}(y_3). \end{aligned}$$

We set

$$\begin{aligned} &h_5(s_1, s_2, s_3, s_4, s_5, y_2, y_3) \\ &:= \mathbf{1}_{0 < s_5 < s_1 < s_4 < s_2 \wedge s_3} \int dy_1 \int dy_4 p(s_1 + s_4 - 2s_5, y_1 - y_4) p(s_2 - s_4, y_2 - y_4) p(s_3 - s_4, y_3 - y_4) \\ &\quad p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1), \end{aligned}$$

and

$$H_5(y_2, y_3) := \int ds_1 \int ds_2 \int ds_3 \int ds_4 \int ds_5 \mathbf{1}_{s < s_2, s_3 < t} h_5(s_1, s_2, s_3, s_4, s_5, y_2, y_3),$$

so that

$$J_5^{\varepsilon,\varepsilon'}(z, z') = \int dy_2 \int dy_3 f_{\varepsilon,z}(y_2) f_{\varepsilon',z'}(y_3) H_5(y_2, y_3).$$

Let us now prove that H_5 is bounded and continuous. Note first that $p(s_1 + s_4 - 2s_5, y_1 - y_4) \leq p(s_1 + s_4 - 2s_5, 0)$. Thus, we easily get

$$\begin{aligned} &h_5(s_1, s_2, s_3, s_4, s_5, y_2, y_3) \\ &\leq \mathbf{1}_{0 < s_5 < s_1 < s_4 < s_2 \wedge s_3} p(s_1 + s_4 - 2s_5, 0) p(s_2 + s_3 - 2s_4, 0) p(s_2 + s_3 - 2s_1, 0). \end{aligned}$$

Now it is easy to check that

$$\begin{aligned} H_5(y_2, y_3) &= \int ds_1 \cdots \int ds_5 \mathbf{1}_{s < s_2, s_3 < t} h_5(s_1, s_2, s_3, s_4, s_5, y_2, y_3) \\ &\leq \int_s^t ds_2 \int_s^t ds_3 \int_0^{s_2 \wedge s_3} ds_4 \int_0^{s_4} ds_1 \int_0^{s_1} ds_5 \\ &\quad p(s_1 + s_4 - 2s_5, 0) p(s_2 + s_3 - 2s_4, 0) p(s_2 + s_3 - 2s_1, 0) = K_5 < \infty. \end{aligned}$$

The function h_5 is continuous and bounded in (y_2, y_3) . From dominated convergence we deduce that H_5 is continuous and bounded. Using that $f_{\varepsilon,z}(y_2) f_{\varepsilon',z'}(y_3) dy_2 dy_3$ converges weakly to $\delta_z(dy_2)\delta_{z'}(dy_3)$, we see that $J_5^{\varepsilon,\varepsilon'}$ tends to

$$(30) \quad J_5^0(z, z') := H_5(z, z') \leq K_5$$

when ε and ε' decrease to 0. Note that $J_5^{\varepsilon,\varepsilon'}(z, z')$ is uniformly bounded by K_5 .

Finally, we study $J_4^{\varepsilon, \varepsilon'}$. Let $g \in \mathcal{B}_+((\mathbb{R}^2)^3)$ and $\bar{g}(x_1, x_2, x_3) := \sum_{\sigma} g(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$, where the sum is over all the permutations σ of $\{1, 2, 3\}$. By a standard monotone class argument we deduce from the moment formula (A.2) for ϱ that

$$\begin{aligned} & \mathbb{E} \left[\int \varrho_v(dy_1) \int \varrho_v(dy_4) \int \varrho_v(dy_5) g(y_1, y_4, y_5) \right] \\ &= 2 i_c \gamma^2 \int_0^v ds_4 \int_{s_4}^v ds_5 \int dy_1 \int dy_4 \int dy_5 \int dy_6 p(v + s_5 - 2s_4, y_1 - y_6) \\ & \quad p(v - s_5, y_4 - y_6) p(v - s_5, y_5 - y_6) \bar{g}(y_1, y_4, y_5) \\ & \quad + i_c^2 \gamma \int_0^v ds_4 \int dy_1 \int dy_4 \int dy_5 p(2v - 2s_4, y_1 - y_4) \bar{g}(y_1, y_4, y_5) \\ & \quad + \frac{1}{3!} i_c^3 \int dy_1 \int dy_4 \int dy_5 \bar{g}(y_1, y_4, y_5). \end{aligned}$$

This implies

$$(31) \quad J_4^{\varepsilon, \varepsilon'} = 2 i_c \gamma^2 J_7^{\varepsilon, \varepsilon'} + i_c^2 \gamma J_8^{\varepsilon, \varepsilon'} + \frac{1}{3!} i_c^3 J_9^{\varepsilon, \varepsilon'},$$

where

$$\begin{aligned} J_7^{\varepsilon, \varepsilon'}(z, z') &:= 2 \int_0^t ds_1 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \int_0^{s_1} ds_4 \int_{s_4}^{s_1} ds_5 \int dy_1 \int dy_4 \int dy_5 \int dy_2 \int dy_3 \int dy_6 \\ & \quad p(s_2 - s_1, y_2 - y_4) p(s_3 - s_1, y_3 - y_5) p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) \\ & \quad f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) \\ & \quad \left[p(s_1 + s_5 - 2s_4, y_1 - y_6) p(s_1 - s_5, y_4 - y_6) p(s_1 - s_5, y_5 - y_6) \right. \\ & \quad + p(s_1 + s_5 - 2s_4, y_4 - y_6) p(s_1 - s_5, y_1 - y_6) p(s_1 - s_5, y_5 - y_6) \\ & \quad \left. + p(s_1 + s_5 - 2s_4, y_5 - y_6) p(s_1 - s_5, y_1 - y_6) p(s_1 - s_5, y_4 - y_6) \right] \end{aligned}$$

and

$$\begin{aligned} J_8^{\varepsilon, \varepsilon'}(z, z') &:= 2 \int_0^t ds_1 \int_0^{s_1} ds_4 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \int dy_1 \int dy_4 \int dy_5 \int dy_2 \int dy_3 p(s_2 - s_1, y_2 - y_4) \\ & \quad p(s_3 - s_1, y_3 - y_5) p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) \\ & \quad \left[p(2s_1 - 2s_4, y_1 - y_4) + p(2s_1 - 2s_4, y_1 - y_5) + p(2s_1 - 2s_4, y_4 - y_5) \right] \end{aligned}$$

as well as

$$\begin{aligned} J_9^{\varepsilon, \varepsilon'}(z, z') &:= 3! \int_0^t ds_1 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \int dy_1 \int dy_4 \int dy_5 \int dy_2 \int dy_3 p(s_2 - s_1, y_2 - y_4) \\ & \quad p(s_3 - s_1, y_3 - y_5) p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3). \end{aligned}$$

We are left to study the convergence of $J_9^{\varepsilon, \varepsilon'}$, $J_8^{\varepsilon, \varepsilon'}$ and $J_7^{\varepsilon, \varepsilon'}$.

First of all, we have

$$\begin{aligned} J_9^{\varepsilon, \varepsilon'}(z, z') &= 3! \int_0^t ds_1 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \int dy_2 \int dy_3 p(s_2 + s_3 - 2s_1, y_2 - y_3) f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) \\ &\leq 6 \int_0^t ds_1 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 p(s_2 + s_3 - 2s_1, 0) =: K_9 < \infty. \end{aligned}$$

As ε and ε' decrease to zero, $J_9^{\varepsilon, \varepsilon'}$ converges to

$$(32) \quad J_9^0(z, z') := 6 \int_0^t ds_1 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 p(s_2 + s_3 - 2s_1, z - z') \leq K_9.$$

Next, we have

$$J_8^{\varepsilon, \varepsilon'}(z, z') = \int dy_2 \int dy_3 f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) H_8(y_2, y_3),$$

where

$$\begin{aligned} H_8(y_2, y_3) &= 2 \int_0^t ds_1 \int_0^{s_1} ds_4 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \\ &\quad \left[\int dy_1 p(s_2 + s_1 - 2s_4, y_1 - y_2) p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) \right. \\ &\quad + \int dy_1 p(s_3 + s_1 - 2s_4, y_3 - y_1) p(s_2 - s_1, y_2 - y_1) p(s_3 - s_1, y_3 - y_1) \\ &\quad \left. + p(s_2 + s_3 - 2s_1, y_2 - y_3) p(s_2 + s_3 - 2s_4, y_3 - y_2) \right]. \end{aligned}$$

Since $p(s_2 + s_1 - 2s_4, y_1 - y_2) \leq p(s_2 + s_1 - 2s_4, 0)$ and $p(s_3 + s_1 - 2s_4, y_3 - y_1) \leq p(s_3 + s_1 - 2s_4, 0)$, we deduce that

$$\begin{aligned} H_8(y_2, y_3) &\leq 6 \int_0^t ds_1 \int_0^{s_1} ds_4 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 p(s_2 + s_1 - 2s_4, 0) p(s_2 + s_3 - 2s_1, 0) \\ &= K_8 < \infty. \end{aligned}$$

Arguments similar to those used for the convergence of $J_5^{\varepsilon, \varepsilon'}$ show that H_8 is continuous and bounded. Thus $J_8^{\varepsilon, \varepsilon'}(z, z')$ is uniformly bounded by K_8 and converges to

$$(33) \quad J_8^0(z, z') := H_8(z, z') \leq K_8.$$

Finally, we have

$$J_7^{\varepsilon, \varepsilon'}(z, z') = \int dy_2 \int dy_3 f_{\varepsilon, z}(y_2) f'_{\varepsilon', z'}(y_3) H_7(y_2, y_3),$$

where

$$\begin{aligned}
H_7(y_2, y_3) := & 2 \int_0^t ds_1 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \int_0^{s_1} ds_4 \int_{s_4}^{s_1} ds_5 \int dy_1 \int dy_6 \\
& \left[p(s_2 - s_5, y_2 - y_6) p(s_3 - s_5, y_3 - y_6) p(s_2 - s_1, y_2 - y_1) \right. \\
& \quad p(s_3 - s_1, y_3 - y_1) p(s_1 + s_5 - 2s_4, y_1 - y_6) \\
& \quad + p(s_2 + s_5 - 2s_4, y_2 - y_6) p(s_3 - s_5, y_3 - y_6) p(s_2 - s_1, y_2 - y_1) \\
& \quad \quad p(s_3 - s_1, y_3 - y_1) p(s_1 - s_5, y_1 - y_6) \\
& \quad + p(s_2 - s_5, y_2 - y_6) p(s_3 + s_5 - 2s_4, y_3 - y_6) p(s_2 - s_1, y_2 - y_1) \\
& \quad \quad \left. p(s_3 - s_1, y_3 - y_1) p(s_1 - s_5, y_1 - y_6) \right].
\end{aligned}$$

Check now that the following upper bound is finite:

$$\begin{aligned}
K_7 := & 2 \int_0^t ds_1 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 \int_0^{s_1} ds_4 \int_{s_4}^{s_1} ds_5 \\
& \left[p(s_2 + s_3 - 2s_5, 0) p(s_2 + s_3 - 2s_1, 0) p(s_1 + s_5 - 2s_4, 0) \right. \\
& \quad + p(s_2 + s_5 - 2s_4, 0) p(s_1 + s_3 - 2s_5, 0) p(s_2 + s_3 - 2s_1, 0) \\
& \quad \left. + p(s_3 + s_5 - 2s_4, 0) p(s_1 + s_2 - 2s_5, 0) p(s_2 + s_3 - 2s_1, 0) \right].
\end{aligned}$$

Arguments similar to those used for the convergence of $J_5^{\varepsilon, \varepsilon'}$ show that H_7 is continuous and bounded by $K_7 < \infty$. Thus $J_7^{\varepsilon, \varepsilon'}(z, z')$ is uniformly bounded by K_7 and converges to

$$(34) \quad J_7^0(z, z') := H_7(z, z') \leq K_7.$$

Altogether, for each $i \in \{1, \dots, 9\}$, $J_i^{\varepsilon, \varepsilon'}$ exists, is uniformly bounded and has a finite limit as $(\varepsilon, \varepsilon') \downarrow 0$. Thus, $J^{\varepsilon, \varepsilon'}(z, z')$ is well-defined and converges in \mathbb{R}_+ as ε and ε' decrease to 0. \square

Completion of the proof of (a). The claimed expectation expression for $\lambda_{[s, t]}(z)$ easily follows from the moment formula (9) for L in the case $m = 1$.

The second moment of $\lambda_{[s, t]}(z)$ is given by the limit J^0 , say, of $J^{\varepsilon, \varepsilon}(z, z)$ from Lemma 9 as $\varepsilon \downarrow 0$. By the formulas (25), (27), (28), and (31),

$$(35) \quad J^0 = 2i_r \kappa \left[2\gamma (2\gamma i_c J_5^0 + i_c^2 J_6^0) + \left\{ 2i_c \gamma^2 J_7^0 + i_c^2 \gamma J_8^0 + \frac{1}{3!} i_c^3 J_9^0 \right\} \right] + i_r^2 J_2^0 < \infty$$

which, in the case $s < t$, is strictly larger than $(\mathbf{E}[\lambda_{[s, t]}(z)])^2$, occurring from the J_2^0 -term [see (26)]. This completes the proof of (a). \square

Remark 10 (variance formula). For $t \geq s \geq 0$ and $z \in \mathbb{R}^d$, from the representation (35) combined with (30), (29), (34), (33), (32), and (26), as well as the expectation formula in (a),

we obtain the following formula for the *variance* of $\lambda_{[s,t]}(z)$:

$$\begin{aligned}
& 2i_c i_r (i_c^2 \kappa + i_r \gamma) \int_0^t ds_1 \int_{s_1 \vee s}^t ds_2 \int_{s_1 \vee s}^t ds_3 p(s_2 + s_3 - 2s_1, 0) \\
& + 8 i_c^2 i_r \gamma \kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2 \vee s}^t ds_3 \int_{s_2 \vee s}^t ds_4 p(s_3 + s_4 - 2s_2, 0) p(s_3 + s_4 - 2s_1, 0) \\
& + 8 i_c^2 i_r \gamma \kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2 \vee s}^t ds_3 \int_{s_2 \vee s}^t ds_4 \int dy p(s_2 + s_3 - 2s_1, y) p(s_3 - s_2, y) \\
& \hspace{15em} p(s_4 - s_2, y) \\
& + 16 i_c i_r \gamma^2 \kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2}^t ds_3 \int_{s_3 \vee s}^t ds_4 \int_{s_3 \vee s}^t ds_5 \int dy_1 \int dy_2 \\
& \hspace{10em} p(s_2 + s_3 - 2s_1, y_1 - y_2) p(s_4 - s_2, y_1) p(s_4 - s_3, y_2) \\
& \hspace{10em} p(s_5 - s_2, y_1) p(s_5 - s_3, y_2) \\
& + 16 i_c i_r \gamma^2 \kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2}^t ds_3 \int_{s_3 \vee s}^t ds_4 \int_{s_3 \vee s}^t ds_5 \int dy_1 \int dy_2 \\
& \hspace{10em} p(s_2 + s_4 - 2s_1, y_2 - y_1) p(s_4 - s_3, y_2) p(s_3 - s_2, y_1) \\
& \hspace{10em} p(s_5 - s_2, y_2 - y_1) p(s_5 - s_3, y_2).
\end{aligned}$$

◇

4.2. Spatial absolute continuity [proof of (b)]. We first prove that,

$$(36) \quad \phi_q(x) \int L(d[r, y]) \mathbf{1}_{[s,t]}(r) p(\varepsilon, x - y),$$

converges in $L^1(\ell \otimes \mathbf{P})$ as ε decreases to 0, to $\phi_q(x)\xi(x)$, where for almost every x , \mathbf{P} -a.s. $\xi = \lambda$. Thanks to the statement (a), it is enough to check that the function

$$(37) \quad (x, \varepsilon) \mapsto \mathbf{E} \left[\int L(d[r, y]) \mathbf{1}_{[s,t]}(r) p(\varepsilon, x - y) \right],$$

is uniformly bounded on $\mathbb{R}^2 \times (0, 1]$. But this is clear since

$$\begin{aligned}
\mathbf{E} \left[\int L(d[r, y]) \mathbf{1}_{[s,t]}(r) p(\varepsilon, x - y) \right] &= \mathbf{E} \left[\int_s^t dr i_r \int dz \int \varrho_r(dy) p(r, z - y) p(\varepsilon, x - y) \right] \\
&= i_r i_c (t - s).
\end{aligned}$$

Statement (b) is then a straight forward consequence of the following criterion with $\nu(dy) = L([s, t], dy)$ [recall Theorem 3 (i)].

Proposition 11 (sufficient criterion for absolute continuity). *Let $\nu \in \mathcal{M}_q$ be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We assume that $\mathcal{E}[(\nu, \phi_q)] < \infty$ and that*

$$(38) \quad \left((x, \omega) \mapsto \phi_q(x) \int \nu(dy) p(\varepsilon, x - y), \quad \varepsilon > 0 \right)$$

converges in $L^1(\ell \otimes \mathcal{P})$ to some $\phi_q \xi$ as $\varepsilon \downarrow 0$. Then \mathcal{P} -a.s. the measure ν is absolutely continuous (with respect to the Lebesgue measure) and has the density function ξ :

$$(39) \quad \nu(dy) = \xi(y)dy.$$

Proof. Let β be any bounded random variable on $(\Omega, \mathcal{F}, \mathcal{P})$, and $f \in \mathcal{B}^q$ continuous. Because of the assumed convergence in $L^1(\ell \otimes \mathcal{P})$, we get that

$$J_\varepsilon := \int dx f(x) \mathcal{E} \left[\beta \int \nu(dy) p(\varepsilon, x-y) \right]$$

converges to $\int dx f(x) \mathcal{E} [\beta \xi(x)]$ as $\varepsilon \downarrow 0$. On the other hand, the function

$$(y, \varepsilon) \mapsto \int dx f(x) p(\varepsilon, x-y)$$

is bounded by $\phi_q(y)$ [thanks to (A.1)], continuous and converges to f as $\varepsilon \downarrow 0$. By dominated convergence, we get that J_ε converges to $\mathcal{E} [\beta (\nu, f)]$. Since β and f are arbitrary, the equality

$$\int dx f(x) \mathcal{E} [\beta \xi(x)] = \mathcal{E} [\beta (\nu, f)]$$

implies that ν is \mathcal{P} -a.s. absolutely continuous with respect to the Lebesgue measure, and that $\nu(dy) = \xi(y)dy$, \mathcal{P} -a.s. \square

4.3. Random ergodic limit [proof of (d)]. Let $f \in L^1_+(\mathbb{R}^2)$. Thanks to Lemma 9, we know that $T^{-1} \int_{[0,T] \times \mathbb{R}^2} L(d[r, y]) f(y)$ is well-defined and even belongs to $L^2(\mathbf{P})$. By self-similarity this has the same law as

$$I_T = T \int_{[0,1] \times \mathbb{R}^2} L(d[r, y]) f(y\sqrt{T}).$$

Thanks to Lemma 9 and (a), we see that I_T converges in $L^2(\mathbf{P})$ to $\lambda_{[0,1]}(0) \int dx f(x)$ as $T \uparrow \infty$. Thus we deduce that for any $f \in L^1_+(\mathbb{R}^2)$, the following convergence in law holds with respect to \mathbb{P} :

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_{[0,T] \times \mathbb{R}^2} L(d[r, y]) f(y) = \lambda_{[0,1]}(0) \int dx f(x).$$

This ends the proof of (d). \square

A. APPENDIX: SOME BASIC PROPERTIES OF CATALYST AND REACTANT

A.1. Moment formulas for the catalyst. Let $d \geq 1$ and fix $\nu \in \mathcal{M}_q$. It is easy to check that for every $T > 0$, there exists a constant $c > 0$ such that for every $x \in \mathbb{R}^d$ and $\varepsilon \in (0, T]$,

$$(A.1) \quad \int dy p(\varepsilon, x-y) \phi_q(y) \leq c \phi_q(x).$$

Therefore we get that if $g \in H_T^q$, then the function $(r, x) \mapsto \int_r^\infty ds P_{s-r}[g(s)](x)$ is well-defined and belongs to H_T^q . If $f \in \mathcal{B}^q$, then the function $(r, x) \mapsto \mathbf{1}_{t \geq r} P_{t-r}[f](x)$ is also well-defined and belongs to H_t^q .

It is well-known that for every $t \geq 0$, $g \in H^q$, $f \in \mathcal{B}^q$, and $m \geq 1$,

$$(A.2) \quad \mathbb{E}_\nu \left[\left[(\varrho_t, f) + \int_0^\infty ds (\varrho_s, g(s)) \right]^m \right] = m! \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{n_1, \dots, n_k \geq 1, \\ n_1 + \dots + n_k = m}} \prod_{i=1}^k (\nu, \chi_{n_i}(0)),$$

where the sequence $(\chi_n, n \geq 1)$ is defined by the recurrence formula

$$(A.3) \quad \chi_n(r, x) := \gamma \int_r^\infty ds \int dy p(s-r, x-y) \left[\sum_{i=1}^{n-1} \chi_i(s, y) \chi_{n-i}(s, y) \right],$$

$(r, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $n \geq 2$, with initial condition

$$(A.4) \quad \chi_1(r, x) := \mathbf{1}_{t \geq r} P_{t-r}[f](x) + \int_r^\infty ds P_{s-r}[g(s)](x), \quad (r, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Thanks to the remark at the beginning of this subsection, we see that the functions χ_n , $n \geq 1$, are well-defined and belong to H^q .

A.2. Regularity properties of the catalyst. We now assume that $d \leq 3$. Recall that we write \mathbb{P} for $\mathbb{P}_{i\ell}$. It is clear from the Hölder continuity Theorem 3 of [DF97a] (p254) that for every $\xi \in (0, 1/4)$, $T \geq 0$, \mathbb{P} -a.s. there exists a constant $C_1 := C(T, \varrho, \xi)$ such that for every $T \geq t \geq r \geq 0$, $f \in \mathcal{B}_+(\mathbb{R}^d)$,

$$(A.5) \quad \int_r^t ds \int \varrho_s(dz) \phi_q(z) f(z) \leq C_1 |t-r|^\xi \int f(z) dz.$$

We have also [cf. Definition 2 b) and Theorem 4 of [DF97a], pp 224 and 259, respectively] that for every $T \geq 0$, $\xi \in (0, 1/4)$, \mathbb{P} -a.s. there exists $C_2 := C(T, \varrho, \xi)$ such that for every $x \in \mathbb{R}^d$, $T \geq t \geq r \geq 0$,

$$(A.6) \quad \int_r^t ds \int \varrho_s(dz) p(s-r, x-z) \phi_q^2(z) \leq C_2 |t-r|^\xi \phi_q(x).$$

A.3. Moment formulas for the reactant. Recall that $d \leq 3$. Using the Markov property of X^ϱ (given ϱ), it is easy to get that \mathbb{P} -a.s. for every $n \geq 1$, $t_n \geq \dots \geq t_1 \geq 0$, and $f_n, \dots, f_1 \in \mathcal{B}_+^q$,

$$(A.7) \quad \mathbb{E}_{r, \mu}^\varrho \left[e^{-\sum_{i \geq r} (X_{t_i}^\varrho, f_i)} \right] = e^{-(\mu, v(r))},$$

where v is the unique nonnegative solution of the catalytic log-Laplace equation (7) with $J(s) := \sum_{t_i \geq s} P_{t_i-s}[f_i]$. Using the continuity of X^ϱ , it can be shown that \mathbb{P} -a.s. for every nonnegative $g \in H^q$,

$$(A.8) \quad \mathbb{E}_{r, \mu}^\varrho \left[e^{-\int_r^\infty ds (X_s^\varrho, g(s))} \right] = e^{-(\mu, v(r))},$$

where v is the unique nonnegative solution of (7) with $J(s) := \int_s^\infty du P_{u-s}[g(u)]$.

We deduce the next result on the *moments of the reactant process* X^ϱ from Theorem 4, Lemma 4 and Remark 2 of [DF97a] (pp 259 and 232, respectively). We have \mathbb{P} -a.s. for every

$t \geq 0$, $g \in H^q$, $f \in \mathcal{B}^q$, and $m \geq 1$,

$$(A.9) \quad E_{r,\mu}^\rho \left[\left[(X_t^\rho, f) + \int_r^\infty ds (X_s^\rho, g(s)) \right]^m \right] = m! \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{n_1, \dots, n_k \geq 1, \\ n_1 + \dots + n_k = m}} \prod_{i=1}^k (\mu, \chi_{n_i}(r)),$$

where $(\chi_n, n \geq 1)$ is defined by the recurrence formula (10) with initial condition

$$(A.10) \quad \chi_1(s, x) := \mathbf{1}_{t \geq s} P_{t-s}[f](x) + \int_s^\infty du P_{u-s}[g(u)](x) \quad (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Since $\chi_1 \in H^q$, inequality (A.6) implies that all the functions χ_n belong to H^q .

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